

Towards efficient algorithm deciding separability of distributed quantum states.

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It is pointed out that separability problem for arbitrary multi-partite states can be fully solved by a finite size, elementary recursive algorithm. In the worse case scenario, the underlying numerical procedure, may grow doubly exponentially with the state's rank. Nevertheless, we argue that for generic states, analysis of concurrence matrices essentially reduces the task of solving separability problem in $m \times n$ dimensions to solving a set of linear equations in about $\binom{mn+D-1}{D}$ variables, where D decreases with mn and for large mn it should not exceed 4. Moreover, the same method is also applicable to multipartite states where it is at least equally efficient.

I. INTRODUCTION

Presence of quantum correlations in a distributed system is probably the most clear and fundamental marker of the system's non-classicality. In applications, quantum correlations make grounds for the emerging field of quantum information technology. In either context, be it fundamental or utilitarian, the notion of quantum entanglement plays the key role [1]. Not surprisingly then, a search for an unambiguous price-tag for the entanglement present in a given arbitrary mixed distributed state has been one of the important tasks of physics of information. It appears, however, that this task is notoriously difficult even in the seemingly simple case of bipartite states. A yet simpler problem of a *finite, operational* necessary and sufficient separability test for bipartite states in arbitrary dimensions is still difficult.

The claim that an arbitrary bipartite $n \otimes m$ state ϱ is separable means that this state can be expressed as a convex combination of projectors on product directions in the bipartite Hilbert space \mathcal{H}_{AB} . This can be formally expressed as the following condition:

The statement

$$\exists(\psi_i^\mu, \phi_j^\mu; i = 1 \dots n, j = 1 \dots m, \mu = 1 \dots (mn)^2 - 1) : \sum_{\mu=1}^{(mn)^2-1} \psi_i^\mu \phi_j^\mu (\psi_\alpha^\mu \phi_\beta^\mu)^* = \varrho_{ij, \alpha\beta} \quad (1)$$

is true.

To decide the truth value of statement (1) is usually difficult, nevertheless it is always possible in principle. The problem to solve is an example of a *decision problem for the existential theory for reals* with $s = (mn)^2 - 1$ (number of real parameters in ϱ) quartic (degree $d = 4$) polynomials in $k = 2(m+n)(m^2n^2 - 1)$ real variables. As such, the problem represents a special case of the *quantifier elimination problem*. The fact that there exist algorithms solving these problems was first proved by A. Tarski [2]. Complexity of his algorithm was, however, not elementary recursive. It means that it could not be bounded from above by any power of exponentials of a finite height.

Although considerable progress in the area has been made over the years and there are working algorithms

for the quantifier elimination problem implemented into such programs like MAPLE or Mathematica, the existing algorithms are far too inefficient to solve non-trivial separability problems. The best general purpose algorithm which we have found is by Basu, Pollack and Roy [3] and it solves the decision problem for the existential theory using $s^{k+1}d^{O(k)}$ arithmetic operations. For separability of a 3×3 system formulated as in statement (1) this means an astronomical number of operations, of the order of $81^{961} \cdot 4^{O(960)} \approx 10^{O(2412)}$.

The line of research aiming at a finite operational (in principle analytic) necessary and sufficient criterion of entanglement in arbitrary distributed states relies on the hope that the head on attack on the problem described above is unnecessarily complicated. Indeed, even for a pair of qubits, solving the decision problem (1) directly would require $16^{121} \cdot 4^{O(120)} \approx 10^{O(220)}$ operations to determine entanglement. One knows, however, that in this case as well as for qubit-qutrit pairs, a simple partial transpose criterion solves the problem efficiently [4]. Moreover, in case of two qubits, Wootters found an efficient finite algorithm to determine an optimal *decomposition* of an arbitrary 2-qubit state [5]. These results suggest some natural approaches to the separability problem in arbitrary dimensions.

One may thus seek for a method which would reduce separability to an eigenvalue problem, like in the partial transposition (PT) test [6]. Among the possible extensions and derivatives of PT, the hierarchy of the PT tests [7] as well as the matrix realignment criterion [8, 9] and its extensions [10, 11] have the desired, operational (ie. eigenvalue-like) form. Moreover, these tests are generally more powerful detectors of entanglement than PT. Nevertheless, they still do not guarantee any universal necessary and sufficient condition for separability. Entanglement witnesses method [4, 12] and its optimizations [13] (important from experimental point of view) have similar drawbacks, although here, an exhaustive search over all the possible pure separable states in principle solves the problem.

To avoid searching over an uncountable set, Gurvits reformulated the separability problem as a weak membership problem [14] and showed that even in this limited

guise, the problem is still NP-hard. In the development which followed, Ioannou et al. proved the existence of an, in principle, efficient algorithm solving the underlying weak membership problem [15]. The structure of the algorithm was, however, complicated, which led to technical difficulties with its implementation.

Hulpke and Bruß [16] showed that irrespective of the required accuracy, the search may be limited to the states with rational coordinates in a given basis. This result combined with the symmetric extensions separability criterion by Doherty et al. [7] guaranteed a finite stopping time (although no upper bound on it) for a separability test on all the states but those exactly on the border between separable and entangled.

Wootters' solution to the separability of two qubits relied on the concept of concurrence and its generalization for mixed states, concurrence matrix [5]. In particular, introduction of concurrence matrix allowed him to derive a simple closed formula for entanglement of formation in terms of the singular values of the matrix. After several attempts (see [17]) the concept of concurrence matrix was extended to arbitrary bipartite systems [18]. There, it appeared that a bilinear combination of concurrence matrices, a *biconcurrence matrix* maybe the fundamental object to investigate in connection with separability of bipartite quantum states. This line of thought was later confirmed by Mintert, Kuś and Buchleitner who used biconcurrence to formulate powerful tests for entanglement of bipartite systems [19]. Recently, the same authors successfully generalized biconcurrence to cover multipartite states as well [20]. This led to powerful tests for different kinds of multi-partite entanglement. However a tight necessary and sufficient operational test still remained out of reach, since it would usually need optimization procedures.

In what follows, we reexamine the role of biconcurrence in solving separability problem for arbitrary distributed states. In particular, we study the underlying decision problem formulated in terms of the eigenvectors of biconcurrence matrix and show that not only for low rank density matrices (like it was in the analysis of range criterion [21, 22]), but also for generic density matrices in many dimensions the resulting solutions are computationally manageable.

In exceptional cases, the resulting numerical calculations may still grow super exponentially with the increasing rank(ϱ). Therefore we do not claim a complete solution to the separability problem. Nevertheless, we can quote evidence that for generic multipartite states the calculations grow polynomially with the size of the investigated system. An improvement of the method which would entirely remove the difficult exceptional cases can hardly be expected since separability problem appears to be computationally hard [14].

II. BIPARTITE SEPARABILITY AND CONCURRENCE.

Entanglement of pure bipartite states is well understood, so let us begin by considering such a state

$$|\psi\rangle_{AB} = \sum_{i,j} a_{ij} |i\rangle_A \otimes |j\rangle_B \quad (2)$$

(the i 's and the j 's here number orthonormal basis vectors in the Hilbert spaces held by Alice (A) and Bob (B) respectively). Separability of $|\psi\rangle_{AB}$ means that the right hand side of (2) factorizes, i.e., that

$$|\psi\rangle_{AB} = \sum_{i,j} f_i |i\rangle_A \otimes g_j |j\rangle_B \quad (3)$$

This is equivalent to the following conditions for the matrix elements a_{ij}

$$\forall(i, j, k, l) \quad a_{ij}a_{kl} - a_{il}a_{kj} = 0 \quad (4)$$

For a pure state in $m \times n$ dimensions, equation (4) imposes $\mathcal{N} = \frac{mn(m-1)(n-1)}{4}$ non-trivial conditions. They correspond to the requirement that all the 2×2 minors of matrix $[a]$ are zero. The degree to which a given pure state violates condition (4) can thus be measured by the length of the \mathcal{N} -dimensional vector \mathbf{C} with components

$$C_\sigma = C_{i \wedge k, j \wedge l} = 2(a_{ij}a_{kl} - a_{il}a_{kj}) \quad (5)$$

This length,

$$C = \sqrt{\sum_{\sigma} |C_\sigma|^2} \quad (6)$$

coincides with concurrence defined for pure multi-dimensional systems by P. Rungta et al. [17].

Separability of a mixed state ϱ means that there exist a decomposition of ϱ , where all the contributing pure states have vanishing concurrence (6). One should notice here that the decomposition of a given state is not unique. In fact, any two ensembles of (sub-normalized) pure states $\{|\psi^\mu\rangle\}$ and $\{|\phi^\mu\rangle\}$ realize the same state $\varrho = \sum_{\mu} |\psi^\mu\rangle\langle\psi^\mu|$ iff they are related by a unitary transformation [23]

$$|\phi^\mu\rangle = \sum_{\nu} U_{\mu\nu} |\psi^\nu\rangle \quad (7)$$

To decide existence of a separable decomposition of a given state is in general difficult. One of the sources of this difficulty may be associated with the fact that the set of pure state concurrences contributing to a mixed state is not closed with respect to the changes of the decomposition. The objects nearest to C which transform well with the decomposition changes are concurrence matrices. For a given state decomposition

$$\varrho = \sum_{\mu} |\psi^\mu\rangle\langle\psi^\mu| \quad (8)$$

with sub-normalized pure components $|\psi^\mu\rangle = \sum_{i,j} a_{ij} |i\rangle_A \otimes |j\rangle_B$, one can define \mathcal{N} symmetric concurrence matrices

$$C_\sigma^{\mu\nu} = a_{ij}^\mu a_{kl}^\nu - a_{il}^\mu a_{kj}^\nu + a_{ij}^\nu a_{kl}^\mu - a_{il}^\nu a_{kj}^\mu \quad (9)$$

When the state decomposition is changed from $\{|\psi^\mu\rangle\}$ to $\{|\phi^\mu\rangle\}$ via transformation (7), then the concurrence matrices $C_\sigma^{\mu\nu}$ change into

$$C_\sigma^{\mu\nu} = \sum_{\alpha\beta} U_{\mu\alpha} C_\sigma^{\alpha\beta} U_{\nu\beta} \quad (10)$$

So, in terms of concurrence matrices, separability means that there exists such a state decomposition for which the diagonals of all the \mathcal{N} concurrence matrices \mathbf{C}_σ are zero. One can immediately notice that in this context two qubits represent an exceptional case: there is only one concurrence matrix. This simplification allowed Wootters to produce an elegant solution of the separability problem for two qubits [5]. Moreover, his solution resulted in a simple and efficient algorithm for the determination of an optimal decomposition of both separable and entangled two-qubit states.

The 2-qubits solution does not, however, generalize easily for systems in higher dimensions. There, in terms of concurrence matrices, a bipartite state is separable iff there exists isometry U such that

$$C_\sigma^{\mu\mu} = \sum_{\alpha\beta} U^{\mu\alpha} C_\sigma^{\alpha\beta} U^{\mu\beta} = 0 \quad (11)$$

for all values of σ simultaneously.

It appears that the question of existence of U which satisfies the whole set of conditions (11) is much more difficult to answer than that of existence of U which satisfies only one of these conditions (for a single given σ) solved in [5]. In particular, in a state with bound entanglement, none of the \mathbf{C}_σ 's will show entanglement separately, nevertheless, regardless of the choice of the local bases, it will be impossible to find a decomposition which zeros the diagonals of all the \mathbf{C}_σ 's together.

To make the problem independent of the local choice of bases and possibly reduce its complexity, one may consider a single object, biconcurrence instead of the set of concurrence matrices [18]. For clarity, one may permute indexes in the original definition (formulae (22) and (23) in [18]) and define biconcurrence \mathbf{B} as

$$B^{\mu\nu;\alpha\beta} = \sum_\sigma C_\sigma^{\mu\nu} C_\sigma^{*\alpha\beta} \quad (12)$$

(the asterisk denotes complex conjugate). Written in this form, \mathbf{B} represents a manifestly positive operator in the vector space of symmetric concurrence matrices.

In terms of biconcurrence, a state ϱ is separable iff given an initial decomposition of the state, e.g., the eigen-decomposition, there exists isometry U such that

$$G_0 = \sum_{i=1} \sum_{\mu\nu\alpha\beta=1} U^{i\mu} U^{i\nu} B^{\mu\nu\alpha\beta} (U^{i\alpha} U^{i\beta})^* = 0 \quad (13)$$

This, together with the conditions making U represent an isometry, allows to associate entanglement with strict positivity of a real quartic form in at most $2mn((mn)^2 - 1)$ variables. The latter problem was formally solved by Jamiolkowski back in 1972 [24]. The level of complexity of this solution was, however, too high to be of practical relevance in the present context.

On the other hand, one may notice that existence of an isometry U solving (13) is equivalent to the existence of such U which zeros the diagonals of all those eigenvectors of B (hereafter denoted by $T_\sigma^{\mu\nu}$), which belong to non-zero eigenvalues. By using biconcurrence in this way, Mintert, Kuś and Buchleitner were able to derive simple, powerful lower bounds on the entanglement of formation for arbitrary bipartite states [19]. In particular, it appeared that unlike the \mathbf{C}_σ 's, single matrices \mathbf{T}_σ can indicate bound entanglement, e.g., in the entangled states introduced in [22] and showed to be non-distillable in [25].

Concurrence matrices can be powerful indicators of entanglement even when it is not seen in any single linear combination of \mathbf{T}_σ matrices, i.e., when the diagonal of every linear combination of \mathbf{T}_σ 's can be brought to zero separately. To see it, it is enough to notice that the equation set (11) (or an equivalent equation set with the \mathbf{T}_σ 's substituted for \mathbf{C}_σ 's) represents a set of up to \mathcal{N} homogeneous quadratic equations for r elements of the μ 'th row of isometry \mathbf{U} and that the elements u_α of each row satisfy the same set of equations (when one begins with the eigendecomposition then r is equal to the state's rank).

$$\sum_{\alpha,\beta=1}^r T_\sigma^{\alpha\beta} u_\alpha u_\beta = 0, \quad \sigma = 1, 2, \dots, \mathcal{N} \quad (14)$$

When the solution set of this equation set is empty then one can clearly claim entanglement right away. Otherwise, whenever the number of equations exceeds $r - 2$, one may expect a finite number of solutions, i.e., a finite number of possible rows of U (modulo normalization). One should expect this to be the prevailing situation since a generic state in $m \times n$ dimensions is of rank $r_g = mn$ while the corresponding number of concurrence matrices is up to $\frac{mn(m-1)(n-1)}{4}$ which is greater than r_g for all the systems greater than two qubits, a qubit and a qutrit and a qubit and a four-level system (a clear indication that the bigger the space, the more exceptional separable states are, as it was suggested by numerical analysis in [26]). To identify the variety of solutions of the equation set (11) is the most costly part of the procedure.

The necessary variable elimination involved in the process and based on a standard construction of a Groebner basis is known to be computationally difficult [27] and even for 0-dimensional ideals, it may be polynomial in d^n , where d is the maximum degree of the generators (original polynomials) and n is the number of variables [28]. This represents a super-exponential growth and,

like a brute force method indicated in the introduction, it may lead to prohibitive calculations already for small systems.

Fortunately, by employing the XL (eXtended Linearization) method [29], one may drastically reduce the level of complexity with growing $\Delta = \mathcal{N} - r + 1$. The method relies on the observation that the set (14) can be regarded as a set of \mathcal{N} linear equations for $\binom{r+1}{2}$ variables $x_{\alpha\beta} = u_\alpha u_\beta$. Moreover, by multiplying each of the equations by all the possible products $u_1^{p_1} \cdots u_r^{p_r}$, such that $\sum_i p_i = D - 2$, one can expand the original set into a homogeneous set of $\mathcal{N} \binom{r+D-3}{D-2}$ linear equations for $\binom{r+D-1}{D}$ variables $x_{\alpha_1, \dots, \alpha_D} = u_{\alpha_1} \cdots u_{\alpha_D}$. The task is now to make the number of equations big enough to eliminate all but the last $D + 1$ variables, e.g., $u_1^D, u_1^{D-1}u_2, \dots, u_2^D$ and put, e.g., $u_1 = 1$, so that one is left with a single polynomial equation in one variable.

Numerical tests of the XL method reported in [29] indicate that for $\Delta = 0$, the degree D of the single variable polynomial necessary to solve in order to obtain the solution set of (14) grows with r like 2^{r-1} . This puts bipartite separability of generic states in 3×3 and even in 4×4 dimensions within a reach of a simple PC. Actually, it may be even simpler than this. In 3×3 dimensions, a generic r is 9 just like a generic \mathcal{N} , thus giving $\Delta = 1$. Numerical tests reported in [29] indicate that in this case $D = r$ which is clearly manageable even for relatively large r . Moreover, according to the same source, for $\Delta = 2$ one has $D \approx \sqrt{r}$ and for $\mathcal{N} = \epsilon r^2$, one should expect $D \approx [1/\sqrt{\epsilon}]$. This indicates that for generic cases in many dimension ($\mathcal{N} \approx r^2/4$) one should not expect the numeric difficulty of the problem to grow substantially with the dimensions of the local Hilbert spaces.

When equation set (14) is solved then, to decide existence of the required isometry, one should arrange the possible rows $(u_{k1}, u_{k2}, u_{k3}, \dots)$ normalized by suitable factors λ_k , so that the resulting matrix \mathbf{U} is an isometry. For that, the factors have to satisfy:

$$\sum_k |\lambda_k|^2 u_{ki} u_{kj}^* = \delta_{ij} \quad (15)$$

This is a simple set of linear equations for $|\lambda_k|^2$ and it can be solved (separability) or proved unsolvable (entanglement) without any problem.

III. BEYOND BIPARTITE STATES.

Our strategy easily generalizes to multipartite states with arbitrary number of parties sharing the state. One only has to analyze eigenvectors of the sum of all the independent positive matrices $\hat{\mathcal{A}}_{jk}^{lm}$ defined in [20] instead of the eigenvectors of a biconcurrence matrix $B^{\mu\nu \alpha\beta}$ discussed earlier. In general, this will increase the number of concurrence matrices to consider, thus making full separability less likely. Everything else goes like for bipartite states. For instance, for a 3-qubit state, there are

three matrices $\hat{\mathcal{A}}$ associated with the three types of possible bi-partite correlations referred to in [20] as $c_1^{(3)}, c_2^{(3)}$ and $c_3^{(3)}$. The maximum possible rank of each of the matrices is three (the symmetric projector is three dimensional). In a generic three-qubit state, one may then expect nine concurrence matrices ($\mathcal{N} = 9$) and $r_g = 8$. To reduce the number of variables to one in the original equation set (14) one will then need such D that $\binom{r+D-1}{D} - D \leq \mathcal{N} \binom{r+D-3}{D-2}$. Here $D = 5$ satisfies the inequality. In general, the bigger the system, the lower D will be sufficient. The difficult part of the algorithm is in the number of the (linear) equation and variables to consider. In the example above, one should expect them to be 1080 and 792 respectively.

IV. CONCLUSIONS.

We have pointed out that separability problem for arbitrary multi-partite states can be fully solved by a finite size, elementary recursive algorithm. However, in the worst case scenario, the underlying numerical procedure may grow super-exponentially with the state's rank.

In an attempt to reduce this complexity, we investigated how analysis of concurrence matrices of given bipartite and even multipartite states may lead to exhaustive analytic separability checks. It appeared that for generic states in $m \times n$ dimensions, analysis of concurrence essentially reduced the task of solving separability problem to solving a set of linear equations in about $\binom{mn+D-1}{D}$ variables, where D decreases with mn and for large mn it should not exceed 4. Moreover, the same method is also applicable to multipartite states where it is at least equally efficient.

One can notice, however, that the relatively low expected complexity of the analysis of the generic states may be nothing more than a reflection of the fact that these states are usually entangled. Indeed, when the variety of the solutions of (14) is either zero-dimensional or empty, then there is not much room for (15) to have any solutions. One possibility to extend the present results can then aim at gaining some understanding of the intermediate case when there are many independent concurrence matrices, nevertheless the variety of solutions of (14) is at least one-dimensional. It would also be interesting to see whether in the language of concurrence it is possible to perform decomposition of the state into a separable part and a low dimensional part like it was for the edge states in optimization process [13]. A possibility like this would realize a version of Lewenstein-Sanpera decomposition [30].

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